THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. MAY, 1897. No. 4.

DISCUSSION OF MERIT CONTESTS IN COLLEGE EXAMINATIONS BY THE METHOD OF LEAST SQUARES.

By CHARLES H. KUMMELL, U. S. Coast and Geodetic Survey, Washington, D. C.

[A paper read before the Philosophical Society of Washington.]

It is well known that it is customary in schools and colleges to estimate merit on the basis of 100 being perfect. If then a number of students 1, 2, 3, ...n receive from judges A, B, C, ...L (number$=m$) the estimates of merit a_1, b_1, c_1, ... l_1; a_2, b_2, c_2, ...l_2; ..... respectively, it is required to find from these discrepant data the most probable merit of each student as well as the personal error of each judge.

Take the case in which three judges A, B, C, have given estimates of merit to seven students, and let the estimates for the first student be:

$a_1=87$; $b_1=70$; $c_1=70$.

Let M_1=true or most probable merit.

Δa_1=error of judge A,

Δb_1=error of judge B,

Δc_1=error of judge C,

then we have undoubtedly,

$$M_1=\tfrac{1}{3}(a_1+b_1+c_1)\pm 0.6745\sqrt{\frac{\Delta a_1^{\,2}+\Delta b_1^{\,2}+\Delta c_1^{\,2}}{3\times 2}}$$

$$=75.7\pm 3.82 \text{ with weight } p_1=1.56.$$

If there was but this one student, then this would be the final answer to the question and the personal errors of the judges must be taken,

$$\Delta a_1 = -11.3,$$

$$\Delta b_1 = +5.7,$$

$$\Delta c_1 = +5.7.$$

If these same judges give estimates of merit to more than one student, we shall have more or less discrepant values of the errors of the judges from which a mean personal error may be determined. Now we have the following individual results for the seven students in the example,

$$M_1 = 87 - 11.3 = 70 + 5.7 = 70 + 5.7 = 75.7 \pm 3.82 \; ; \; p_1 = 1.56,$$

$$M_2 = 92 - 4.3 = 76 + 11.7 = 95 - 7.3 = 87.7 \pm 3.98 \; ; \; p_2 = 1.44,$$

$$M_3 = 80 - 11.7 = 60 + 8.3 = 65 + 3.3 = 68.3 \pm 4.05 \; ; \; p_3 = 1.38,$$

$$M_4 = 93 - 8.3 = 68 + 16.7 = 93 - 8.3 = 84.7 \pm 5.62 \; ; \; p_4 = 0.71,$$

$$M_5 = 85 - 11.7 = 67 + 6.3 = 68 + 5.3 = 73.3 \pm 3.94 \; ; \; p_5 = 1.47,$$

$$M_6 = 95 - 7.3 = 78 + 9.7 = 90 - 2.3 = 87.7 \pm 3.40 \; ; \; p_6 = 1.98,$$

$$M_7 = 96 - 9.0 = 80 + 7.0 = 85 + 2.0 = 87.0 \pm 3.19 \; ; \; p_7 = 2.24.$$

In examining these results we notice that judge A always over-estimates, judge B under-estimates, and that judge C is the least consistent of the three judges, as can be roughly seen from their ranges, being 7.4 for A, 11.0 for B, and 14.0 for C. To determine the personal errors of the judges, the arithmetical mean of their errors might be taken, but it is more rigorous to take their weighted mean, using the above weights, which are reciprocally proportional to the squares of the probable errors. We have thus,

$$\Delta a = \frac{[p\Delta a]}{[p]} \pm 0.6745 \sqrt{\frac{[p\Delta a^2] - [p]\Delta a^2}{[p](n-1)}}$$

=personal error of judge A, and similarly for the other judges. We then have the numerical results :

$$\Delta a = -9.0 \pm 0.70 \; ; \; P_a = 1.53.$$

$$\Delta b = +8.7 \pm 0.79 \; ; \; P_b = 1.20.$$

$$\Delta c = +0.4 \pm 1.31 \; ; \; P_c = 0.44.$$

It is obvious that if we correct the original estimates by these quantities, the corrected merits by judge A will be nearest the truth, those of B will be next best, and those of C will hardly be improved ; hence the merits must be redetermined by the formula,

$$M_r = a_r + \Delta a + \Delta^2 a_r = b_r + \Delta b + \Delta^2 b_r = c_r + \Delta c + \Delta^2 c_r,$$

$$= \frac{P_a(a_r + \Delta a) + P_b(b_r + \Delta b) + P_c(c_r + \Delta c)}{P_a + P_b + P_c}$$

$$\pm 0.6745 \sqrt{\frac{P_a(\Delta a^2 + \Delta^2 a_r^2) + P_b(\Delta b^2 + \Delta^2 b_r^2) + P_c(\Delta c^2 + \Delta^2 c_r^2)}{(P_a + P_b + P_c)(3-1)}}$$

which gives the following numerical results :

$M_1 = 78.0 - 0.8 = 78.7 - 1.5 = 70.4 + 6.8 = 77.2 \pm 4.14,$

$M_2 = 83.0 + 2.4 = 84.7 + 0.7 = 95.4 - 10.0 = 85.4 \pm 4.33,$

$M_3 = 71.0 - 1.7 = 68.7 + 0.6 = 65.4 + 3.9 = 69.3 \pm 4.03,$

$M_4 = 84.0 - 1.5 = 76.7 + 5.8 = 93.4 - 10.9 = 82.5 \pm 4.72,$

$M_5 = 76.0 - 1.2 = 75.7 - 0.9 = 68.4 + 6.4 = 74.8 \pm 4.12,$

$M_6 = 86.0 + 0.9 = 86.7 + 0.2 = 90.4 - 3.5 = 86.9 \pm 3.98,$

$M_7 = 87.0 + 0.4 = 88.7 - 1.3 = 85.4 + 2.0 = 87.4 \pm 3.97.$

These results show, as they should, that the corrected estimates of judge *A* came nearest to the correct value, *B's* next best, and *C's* hardly improved ; their ranges being 4.1, 7.3 and 17.7 respectively. We also notice that now contestant 7 has the highest merit while in the first approximation 2 and 6 came out best with a tie. The reason for this is that 2 and 6 received very high marks from judge *C*, which have very small weight in the second approximation. There is an apparent paradox in this, that the best values of the merits have nevertheless larger probable errors than those of the first approximation. Now each of the arithmetic means of the first approximation involves only three errors out of the 27. By correcting the estimates by the personal errors of the judges, a secondary effect of the remaining 24 errors is added in each case.

The second approximation, although sufficiently close to the true values may however, yet be improved. For we now have more correct actual errors of the judges as follows :

$\Delta a_1 = -9.8$; $\Delta b_1 = +7.2$; $\Delta c_1 = +7.2,$

$\Delta a_2 = -6.6$; $\Delta b_2 = +9.4$; $\Delta c_2 = -9.6,$

$\Delta a_3 = -10.7$; $\Delta b_3 = +9.3$; $\Delta c_3 = +4.3,$

$\Delta a_4 = -10.5$; $\Delta b_4 = +14.5$; $\Delta c_4 = -10.5,$

$\Delta a_5 = -10.2$; $\Delta b_5 = +7.8$; $\Delta c_5 = +6.8,$

$\Delta a_6 = -8.1$; $\Delta b_6 = +8.9$; $\Delta c_6 = -3.1,$

$\Delta a_7 = -8.6$; $\Delta b_7 = +7.4$; $\Delta c_7 = +2.4.$

The weighted means of these, weights being given according to their probable errors, will be new values of the personal errors Δa, Δb, Δc of the judges and applying these we obtain new values for the merits. It is easily seen however that this can only affect the 0.01 and though theoretically speaking an infinite number of approximations is required to obtain the most probable values of the merits we may safely regard the second approximation as sufficient.

ON THE CIRCULAR POINTS AT INFINITY.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

I. THE COORDINATE SYSTEM. In the following discussion, in addition to the usual Cartesion coördinates, homogeneous point and line coördinates will be used. They are related to Cartesian coördinates as follows :*

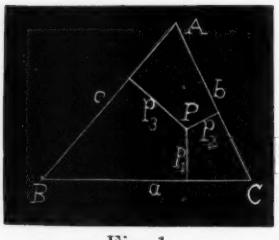


Fig. 1.

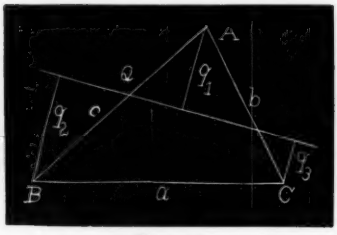


Fig. 1'.

In figure 1, p_1, p_2, p_3 are the perpendicular distances of a point P, from the sides of the coördinate triangle.

In figure 1', q_1, q_2, q_3 are the perpendicular distances of a line Q, from the vertices of the coördinate triangle.

The three point coördinates of P are expressed as follows :

$$\begin{aligned} \rho x_1 &= p_1 \varkappa_1 \\ \rho x_2 &= p_2 \varkappa_2 \\ \rho x_3 &= p_3 \varkappa_3 \end{aligned} \qquad (1)$$

The three line coördinates of Q are expressed as follows :

$$\begin{aligned} \sigma u_1 &= q_1 \lambda_1 \\ \sigma u_2 &= q_2 \lambda_2 \\ \sigma u_3 &= q_3 \lambda_3 \end{aligned} \qquad (1')$$

The $\varkappa$'s and λ's are six constants which might be chosen at pleasure, but for convenience are chosen in a particular way.

*For a fuller treatment see Clebsch, Vorlesungen ueber Geometrie, S. 27-29, S. 62-78.

In Cartesian

point coördinates the equations of the sides of the triangle may be

$$\begin{aligned} a_1x+b_1y+c_1&=0 \text{ for } BC \\ a_2x+b_2y+c_2&=0 \text{ for } CA \\ a_3x+b_3y+c_3&=0 \text{ for } AB \end{aligned} \qquad (2)$$

line coördinates the equations of the opposite vertices will be

$$\begin{aligned} A_1u+B_1v+C_1&=0 \text{ for } A \\ A_2u+B_2v+C_2&=0 \text{ for } B \\ A_3u+B_3v+C_3&=0 \text{ for } C \end{aligned} \qquad (2)'$$

We have then

$$p_1=\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} \qquad q_1=\frac{A_1u+B_1v+C_1}{C_1\sqrt{u^2+v^2}}$$

$$p_2=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad (3) \qquad q_2=\frac{A_2u+B_2v+C_2}{C_2\sqrt{u^2+v^2}} \qquad (3)'$$

$$p_3=\frac{a_3x+b_3y+c_3}{\sqrt{a_3^2+b_3^2}} \qquad q_3=\frac{A_3u+B_3v+C_3}{C_3\sqrt{u^2+v^2}}$$

We choose

$$\kappa_1=\sqrt{a_1^2+b_1^2},\ \kappa_2=\sqrt{a_2^2+b_2^2},\ \kappa_3=\sqrt{a_3^2+b_3^2} \qquad \lambda_1=C_1,\ \lambda_2=C_2,\ \lambda_3=C_3$$

and write

$$\begin{aligned} \rho x_1&=a_1x+b_1y+c_1 \\ \rho x_2&=a_2x+b_2y+c_2 \\ \rho x_3&=a_3x+b_3y+c_3 \end{aligned} \qquad (A) \qquad \begin{aligned} \sigma u_1&=A_1u+B_1v+C_1 \\ \sigma u_2&=A_2u+B_2v+C_2 \\ \sigma u_3&=A_3u+B_3v+C_3 \end{aligned} \qquad (A)'$$

Solving these equations, and writing r for (abc),

$$x=\frac{\rho(A_1x_1+A_2x_2+A_3x_3)}{r} \qquad u=\frac{\sigma r(a_1u_1+a_2u_2+a_3u_3}{r^2}$$

$$y=\frac{\rho(B_1x_1+B_2x_2+B_3x_3)}{r} \qquad v=\frac{\sigma r(b_1u_1+b_2u_2+b_3u_3}{r^2}$$

$$1=\frac{\rho(C_1x_1+C_2x_2+C_3x_3}{r} \qquad 1=\frac{\sigma r(c_1u_1+c_2u_2+c_3u_3}{r^2}$$

or

$$x=\frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3} \qquad u=\frac{a_1u_1+a_2u_2+a_3u_3}{c_1u_1+c_2u_2+c_3u_3}$$

$$(5) \qquad (5)'$$

$$y=\frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3} \qquad v=\frac{b_1u_1+b_2u_2+b_3u_3}{c_1u_1+c_2u_2+c_3u_3}$$

Upon our choice of the κ's and λ's depends the following important result :

$$\rho\sigma(u_1x_1+u_2x_2+u_3x_3)=r(ux+vy+1). \tag{6}$$

Hence $u_1x_1+u_2x_2+u_3x_3$ vanisnes whenever $ux+vy+1$ vanishes. We may note that

$C_1x_1+C_2x_2+C_3x_3=0$ is the equation of the line at infinity. It gives the condition that $x=y=\infty$. (7)

$c_1u_1+c_2u_2+c_3u_3=0$, is the equation of the origin of coördinates. It gives the condition that $u=v=\infty$. (7)′

II. The Circular Points.*

A. Proof that all circles whatsoever pass through two points at infinity.

The equations of any two circles may be written,

$$\begin{aligned} x^2+y^2+2gx+2fy+c&=0.\\ x^2+y^2+2g'x+2f'y+c'&=0. \end{aligned} \tag{8}$$

In homogeneous point coördinates,

$$x=\frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{A}{C}$$

by (5) and our equations become,

$$y=\frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{B}{C}.$$

$$\begin{aligned} A^2+B^2+2gAC+2fBC+cC^2&=0.\\ A^2+B^2+2g'AC+2f'BC+c'C^2&=0. \end{aligned} \tag{9}$$

The lines passing through their points of intersection are, by subtraction,

$$C[2(g-g')A+2(f-f')B+(c-c')C]=0. \tag{10}$$

Of these the line $C=0$, is the one that interests us. It is the equation of the line at infinity. The infinite points are found by solving $C=0$, with the equation of either circle, and thus we find them from $C=0$, and $A^2+B^2=0$, or in Cartesian coördinates from the equation of the line at infinity and $x^2+y^2=0$; this would give the same solution always for any two circles; therefore every circle passes through two points at infinity.

B. Cartesian equation of the points in line coördinates.

The equations of the points in Cartesian line coördinates may be readily obtained.

*See Clebsch, Vorlesungen ueber Geometrie, S. 145-149. Salmon's Conic Sections, pages 238, 325. Fiedler's Salmon, Analytische Geometrie der Kegelschnitte, S. 208.

As $x=\frac{A}{C}$, $y=\frac{B}{C}$, the equation of a point $ux+vy+1=0$, becomes

$$Au+Bv+C=0.$$

$x+yi=0$, gives for any point on the line $A+Bi=0$.
We must then have for one of the points $Au+Bv+C=0$, all true.

$$A+Bi=0$$
$$C=0$$

Hence $\begin{vmatrix} u & v & 1 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{vmatrix}=0$, or $ui-v=0$,

similarly for the other point $-ui-v=0$.

For the pair we have $u^2+v^2=0$. (11)

C. Coördinates of the circular points.

1. *Homogeneous rectangular coördinates.* We saw that we could find the coördinates of the circular points by solving the equations of the line at infinity, $x^2+y^2=0$. The equation of the line at infinity is $0x+0y+c=0$. $x^2+y^2=(x+yi)(x-yi)=0$, a pair of imaginary straight lines through the origin. We will find the intersections of a line $ax+by+c=0$ with $x+yi=0$, and $x-yi=0$.

Solving $ax+by=-c$, we get $x=\frac{-ci}{ai-b}$.

$$x+yi=0 \qquad y=\frac{c}{ai-b}.$$

Or $\frac{x}{-i}=\frac{y}{1}=\frac{c}{ai-b}$. If now we introduce a third coördinate c to make our rectangular coöordinates homogeneous, and consider the ratios of x, y, and c as coördinates, we see that the coördinates of one imaginary circular point are given by $x:y:c=-i:1:0$,
and of the other by $x:y:c=i:1:0$. (12)

2. *Homogeneous point coöordinates.* The coördinates of the circular points however assume a more convenient form when expressed in the general point coödinates. We shall obtain them in proving the following highly interesting proposition:

A circle, with fixed center in the finite region, whose radius becomes indefinitely great, degenerates into the two circular points at infinity.

We will obtain the equation of the circle in homogeneous line coördinates. We express that a line u is always at a distance r from the point x', the center of the circle.

Let $u_1x_1+u_2x_2+u_3x_3=0$ be the point equation of the line [see (6)].
This by (4) is

$$u_1(a_1x+b_1y+c_1)+u_2(a_2x+b_2y+c_2)+u_3(a_3x+b_3y+c_3)=0$$

or $(u_1a_1+u_2a_2+u_3a_3)x+(u_1b_1+u_2b_2+u_3b_3)y+(u_1c_1+u_2c_2+u_3c_3)=0.$ (13)

Putting this in cosine form, and taking the square of the distance from $x'y'$ to the line equal to r^2, we get,

$$\frac{[(u_1a_1+u_2a_2+u_3a_3)x'+(u_1b_1+u_2b_2+u_3b_3)y'+(u_1c_1+u_2c_2+u_3c_3)]}{(u_1a_1+u_2a_2+u_3a_3)^2+(u_1b_1+u_2b_2+u_3b_3)^2}=r^2.$$

or (14)*

$$\frac{(u_1x_1'+u_2x_2'+u_3x_3')^2}{\kappa_1^2u_1^2+\kappa_2^2u_2^2+\kappa_3^2u_3^2-2\kappa_1\kappa_2u_1u_2\cos C-2\kappa_2\kappa_3u_2u_3\cos A-2\kappa_3\kappa_1u_3u_1\cos B}=r^2$$

The reductions in the denominator depend on the following:

$$a_1^2+b_1^2=\kappa_1^2, \text{ etc.} \quad a_1a_2+b_1b_2=\kappa_1\kappa_2\left(\frac{a_1b_2}{\kappa_1\kappa_2}+\frac{b_1b_2}{\kappa_1\kappa_2}\right)$$

$$=\kappa_1\kappa_2(\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2)=\kappa_1\kappa_2\cos C, \text{ etc.},$$

where α_1, α_2, α_3 are the angles that p_1, p_2, p_3 make with the axis of x, and the origin is taken within the coördinate triangle. (14) is the line equation of the circle. We notice now that the expression in the denominator may be factored, for considering the variables as, κ_1u_1, κ_2u_2, κ_3u_3, its discriminant is

$$\begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix}$$

and that this is zero, may be shown as follows:

From trigonometry we have the three equations:

$$\begin{aligned} a-b\cos C-c\cos B&=0 \\ -a\cos C+b-c\cos A&=0 \\ -a\cos B-b\cos A+C&=0 \end{aligned}$$

whence it follows that the determinant of the coefficients of a, b, and c, vanishes. Put $\kappa_1u_1=l$, etc. Then

$$l^2+m^2+n^2-2lm\cos C-2mn\cos A-2nl\cos B=(l\alpha+m\beta+n\gamma)(l\alpha'+m\beta'+n\gamma')$$

$$=l^2\alpha\alpha'+m^2\beta\beta'+n^2\gamma\gamma'+lm(\alpha\beta'+\alpha'\beta)+mn(\beta\gamma'+\beta'\gamma)+nl(\alpha\gamma'+\alpha'\gamma),$$

and we must have:

*Compare Salmon's Conic Sections, page 128, Ex. 6.

$$\begin{array}{ll} \alpha\alpha'=1. & \text{Take } \alpha=\cos B-i\sin B, \quad \gamma=-1, \\ \beta\beta'=1. & \qquad\quad \alpha'=\cos B+i\sin B, \quad \gamma'=-1, \\ \gamma\gamma'=1. & \qquad\quad \beta=\cos A+i\sin A, \\ & \qquad\quad \beta'=\cos A-i\sin A, \end{array}$$

and the first three conditions are satisfied. Also,

$$\alpha\beta'+\alpha'\beta=[\cos(A+B)-i\sin(A+B)+\cos(A+B)+i\sin(A+B)]$$
$$=2\cos(A+B)=-2\cos C.$$

$$\beta\gamma'+\beta'\gamma=-\cos A-i\sin A-\cos A+i\sin A=-2\cos A.$$

$$\alpha\gamma'+\alpha'\gamma=-\cos B+i\sin B-\cos B-i\sin B=-2\cos B.$$

Our expression therefore has the two factors, viz :

$$[(\cos B-i\sin B)\varkappa_1 u_1+(\cos A+i\sin A)\varkappa_2 u_2-\varkappa_3 u_3]=L.$$

$$[(\cos B+i\sin B)\varkappa_1 u_1+(\cos A-i\sin A)\varkappa_2 u_2-\varkappa_3 u_3]=M.$$

Also write $u_1x_1+u_2x_2+u_3x_3=u_x$, then our line equation of the circle may be written

$$LM=\left(\frac{u_{x'}}{r}\right)^2. \qquad (15)$$

If γ becomes indefinitely great $u_{x'}$ does not become indefinitely great, for the x''s are finite, the coördinates of the fixed center, and the u's by $(4)'$ are always finite, since u and v are always finite, and for a line which is moved off to infinity approach zero together. It follows that $\lim_{r=\infty}\left(\frac{u_{x'}}{r}\right)=0$.

Hence the equation of a circle whose radius is infinite, and whose center is in the finite region is in line coördinates,

$$LM=0. \qquad (16)$$

But this is also the equation of a point-pair, and since we have proved that every circle whatsoever contains the two imaginary circular points at infinity, it follows that the two points into which this circle has degenerated are themselves the two imaginary circular points at infinity. As we might just as well have factored our expression LM in two other ways, in which the two angles B and C, or C and A, play the same part as A and B, we may write the coördinates of the two points in the three ways, as follows :

$$\begin{array}{lllll} (1). & -\varkappa_1, & \varkappa_2 e^{-iC}, & \varkappa_3 e^{iB}. & \\ & -\varkappa_1, & \varkappa_2 e^{iC}, & \varkappa_3 e^{-iB}. & \\ (2). & \varkappa_1 e^{iC}, & -\varkappa_2, & \varkappa_3 e^{-iA}. & (17). \\ & \varkappa_1 e^{-iC}, & -\varkappa_2, & \varkappa_3 e^{iA}. & \\ (3). & \varkappa_1 e^{-iB}, & \varkappa_2 e^{iA}, & -\varkappa_3. & \\ & \varkappa_1 e^{iB}, & \varkappa_2 e^{-iA}, & -\varkappa_3. & \end{array}$$

Before going on to the next division of our discussion, we will recall* that if

$a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+$
$2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{31}x_3x_1=0$
where $a_{ik}=a_{ki}$ is the point equation of a conic, with non-vanishing discriminant,

$A_{11}u_1^2+A_{22}u_2^2+A_{33}u_3^2+$
$2A_{12}u_1u_2+2A_{23}u_2u_3+2A_{31}u_3u_1=0$
where $A_{ik}=A_{ki}$ is the line equation of a conic, with non-vanishing discriminant,

then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & u_1 \\ a_{21} & a_{22} & a_{23} & u_2 \\ a_{31} & a_{32} & a_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0. \qquad (18)$$

is the line equation of the same conic.

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & x_1 \\ A_{21} & A_{22} & A_{23} & x_2 \\ A_{31} & A_{32} & A_{33} & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = 0. \qquad (18)'$$

is the point equation of the same conic.

and that always

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & x_1 & x_1' \\ a_{21} & a_{22} & a_{23} & x_2 & x_2' \\ a_{31} & a_{32} & a_{33} & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix} = 0 \qquad (19)$$

is the equation of the pair of tangents from the point x' to the same conic.

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & u_1 & u_1' \\ A_{21} & A_{22} & A_{23} & u_2 & u_2' \\ A_{31} & A_{32} & A_{33} & u_3 & u_3' \\ u_1 & u_2 & u_3 & 0 & 0 \\ u_1' & u_2' & u_3' & 0 & 0 \end{vmatrix} = 0 \qquad (19)'$$

is the equation of the pair of points where the line u' cuts the same conic.

III. Fundamental Geometrical Relations Defined in Terms of the Circular Points at Infinity.

A. The equation of a circle in terms of the circular points.

Our line equation of the circle (14) may be written:

$$(r^2\kappa_1^2-x_1^2)u_1^2+(r^2\kappa_2^2-x_2'^2)u_2^2+(r^2\kappa_3^2-x_3'^2)u_3^2$$
$$-2(r^2\kappa_1\kappa_2\cos B+x_1'x_2')u_1u_2-2(r^2\kappa_2\kappa_3\cos A+x_2'x_3')u_2u_3$$
$$-2(r^2\kappa_3\kappa_1\cos B+x_3'x_1')u_3u_1=0.$$

Therefore by (18)', its point equation is,

$$\begin{vmatrix} \gamma^2\kappa_1^2-x_1'^2 & -(\gamma^2\kappa_1\kappa_2\cos C+x_1'x_2') & -(\gamma^2\kappa_2\kappa_3\cos B+x_1'x_3') & x_1 \\ -(r^2\kappa_1\kappa_2\cos C+x_1'x_2') & r^2\kappa_2^2-x_1'^2 & -(r^2\kappa_2\kappa_3\cos A+x_2'x_3') & x_2 \\ -(r^2\kappa_1\kappa_3\cos B+x_1'x_3') & -(r^2\kappa_2\kappa_3\cos A+x_2'x_3') & r^2\kappa_3^2-x_3'^2 & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = 0.$$

The coefficients of x_1^2 and x_1x_2 changed in sign will be:

*Clebsch, Vorlesungen ueber Geometrie, S. 113.

Of x_1^2, $(r^2\kappa_2^2-x_2'^2)(r^2\kappa_3^2-x_3'^2)-(r^2\kappa_2\kappa_3\cos A+x_2'x_3')^2$

$$=r^4\kappa_2^2\kappa_3^2-r^2(\kappa_3^2x_2'^2+\kappa_2^2x_3'^2)-r^4\kappa_2^2\kappa_3^2\cos^2A-2r^2\kappa_2\kappa_3x_2'x_3'\cos A,$$

of x_1x_2, $2(r^2\kappa_3^2-x_3'^2)(r^2\kappa_1\kappa_2\cos C+x_1'x_2')$

$$+2(r^2\kappa_2\kappa_3\cos A+x_2'x_3')(r^2\kappa_1\kappa_3\cos B+x_1'x_3')$$

$$=2(r^4\kappa_1\kappa_2x_3^2\cos C-r^2\kappa_1\kappa_2\cos Cx_3'^2+r^2\kappa_3^2x_1'x_2'+r^4\kappa_1\kappa_2\kappa_3\cos A\cos B$$
$$+r^2\kappa_2\kappa_3x_1'x_3'\cos A+r^2\kappa_1\kappa_3x_2'x_3'\cos B).$$

Similarly for the other terms. Put terms containing r^2 on right hand side of equation, divide by r^2, arrange, and reduce, and we get finally,

$$r^2(\kappa_2\kappa_3\sin Ax_1+\kappa_3\kappa_1\sin Bx_2+\kappa_1\kappa_2\sin Cx_3)^2$$

$$=\kappa_3^2(x_2'x_1-x_2x_1')^2+\kappa_1^2(x_3'x_2-x_3x_2')^2+\kappa_2^2(x_1'x_3-x_1x_3')^2$$

$$-2\kappa_2\kappa_3(x_3x_1'-x_3'x_1)(x_1x_2'-x_1'x_2)\cos A-2\kappa_3\kappa_1(x_1x_2'-x_1'x_2)(x_2x_3'-x_2'x_3)\cos B$$

$$-2\kappa_1\kappa_2(x_2x_3'-x_2'x_3)(x_3x_1'-x_3x_1)\cos C.* \qquad (20)$$

From what we did with LM of (14) it is clear that the right member of (20) can be factored. Put $(x_3'x_2-x_3x_2')=l$, etc. We then have as factors,

$$l\kappa_1e^{-iB}+m\kappa_2e^{iA}-n\kappa_3 \text{ or if } \rho\xi_1=\kappa_1e^{-iB},\ \rho\xi_1'=\kappa_1e^{iB}.$$
$$l\kappa_1e^{iB}+m\kappa_2e^{-iA}-n\kappa_3 \qquad \rho\xi_2=\kappa_2e^{iA},\ \rho\xi_2'=\kappa_2e^{-iA}.$$
$$\rho\xi_3=-\kappa_3,\ \rho\xi_3'=-\kappa_3.$$

The factors become,

$$\rho(l\xi_1+m\xi_2+n\xi_3)$$
$$\rho(l\xi_1'+m\xi_2'+n\xi_3')$$

and by supplying the values of l, m, and n, these become

$$\rho\begin{vmatrix}\xi_1 & x_1 & x_1'\\ \xi_2 & x_2 & x_2'\\ \xi_3 & x_3 & x_3'\end{vmatrix}, \qquad \rho\begin{vmatrix}\xi_1' & x_1 & x_1'\\ \xi_2' & x_2 & x_2'\\ \xi_3' & x_3 & x_3'\end{vmatrix}$$

Also the left member is $r^2(C_1x_1+C_2x_2+C_3x_3)^2$ since

$$\kappa_2\kappa_3\sin A=\kappa_2\kappa_3\sin(\alpha_3-\alpha_2)=\kappa_2\kappa_3\left(\frac{a_2b_3-a_3b_2}{\kappa_2\kappa_3}\right)=C_1, \text{ etc.},$$

*Compare Salmon's Conic Sections, page 128, Ex. 6.

and (20) takes the form

$$r^2(C_1x_1+C_2x_2+C_3x_3)^2=\rho^2(\xi xx')(\xi' xx'). \qquad (21)$$

As a check on our determination of the coördinates of the circular points at infinity, let us see if the coördinates, $\rho\xi$, $\rho\xi'$ satisfy this equation of the circle. They certainly reduce the right member to zero, for $(\xi\xi x')=0$, and $(\xi'\xi' x')=0$. They also reduce the left member to zero, for it is reduced to zero, by the coördinates of any infinitely distant points [see (7)]. We thus confirm the results of (17), and (21) is the equation of a circle, whose center is at x', in terms of the coördinates of the circular points at infinity.*

B. General distance formula in terms of the circular coördinates.

The preceding result (21) may be used as a formula for the distance between two points x and x'. It must first however be made homogeneous in all the coördinates. It is clear that,

$$C_1x_1+C_2x_2+C_3x_3=\varkappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ x_2 & \xi_2 & \xi_2' \\ x_3 & \xi_3 & \xi_3' \end{vmatrix},$$

for the vanishing of both expressions signifies a line through two infinitely distant points. To determine $\varkappa$, let $x_2=x_3=0$.

Then $C_1x_1=\varkappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ 0 & \xi_2 & \xi_2' \\ 0 & \xi_3 & \xi_3' \end{vmatrix}$

$$=\varkappa\rho^2x_1\frac{(-\varkappa_2\varkappa_3(\cos A+i\sin A)+\varkappa_2\varkappa_3(\cos A-i\sin A)}{\rho^2}$$

$$=-2\varkappa x_1\varkappa_2\varkappa_3 i\sin A$$

$$=-2\varkappa x_1 iC_1.$$

$\therefore\ \varkappa=-\dfrac{1}{2i}=\frac{1}{2}i$, and we have

$$C_1x_1+C_2x_2+C_3x_3=\tfrac{1}{2}(i\rho^2)(x\xi\xi')=$$

a constant by the first solution of (4).

By this fact we can make r^2 homogeneous. For

$$r^2=c\frac{(\xi xx')(\xi' xx')}{(x\xi\xi')^2(x'\xi\xi)^2}$$

where c is some constant. To determine it let the distance between a and a' be

*A question might arise as to the constant ρ^2. That can be disposed of as is done in the next division.

unity. Substituting the value of ρ so obtained we have

$$r^2 = \frac{(\xi xx')(\xi' xx')(a\xi\xi')^2(a'\xi\xi')^2}{(x\xi\xi')^2(x'\xi\xi')^2(\xi aa')(\xi' aa')}, \quad (22)$$

which is seen to be homogeneous and of degree zero in all the coördinates. It is also clear that it is an absolute invariant expression in ternary forms, for on account of the multiplication law of determinants any determinant of the form $(x'y'z')$ in terms of the new variables becomes equal under linear transformation, to $M(xyz)$ where M is the modulus of the transformation, and the transformation equations are:

$$\begin{aligned}
\rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\
\rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\
\rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \ *
\end{aligned}$$

Our distance is thus defined projectively with respect to the circular points.

C. The equation of the pair of lines from a point x to the two circular points.

These lines will certainly be given by $(xx'\xi)(xx'\xi')=0$. (23).

By (16) the equation of the circular points in line coördinates is

$$\kappa_1^2 u_1^2 + \kappa_2 u_2^2 + \kappa_3 u_3^2 - 2\kappa_1\kappa_2 u_1 u_2 \cos C - 2\kappa_2\kappa_3 u_2 u_3 \cos A - 2\kappa_3\kappa_1 u_3 u_1 \cos B = 0.$$

Now the equation of the pair of tangents from x', to the points will be by (19),

$$\begin{vmatrix}
\kappa_1^2 & -\kappa_1\kappa_2\cos C & -\kappa_1\kappa_3\cos B & x_1 & x_1' \\
-\kappa_1\kappa_2\cos C & \kappa_2^2 & -\kappa_2\kappa_3\cos A & x_2 & x_2' \\
-\kappa_1\kappa_3\cos B & -\kappa_2\kappa_3\cos A & \kappa_3^2 & x_3 & x_3' \\
x_1 & x_2 & x_3 & 0 & 0 \\
x_1' & x_2' & x_3' & 0 & 0
\end{vmatrix} = 0. \quad (24)$$

It follows that this determinant is equal to the left member of (23) multiplied by a constant, since their vanishing represents the same geometrical form. Let κ denote this constant. Put $x_1 = a_4$, $x_2' = b_5$, above. $\kappa_3^2 = c_3$, $x_1 = d_1$, $x_2' = e_2$, below. On the left hand the term containing $x_1^2 x_2'^2$ will be represented by $a_4 b_5 c_3 d_1 e_2$. The number of inversions of order $j = 3+3+2 = 8$. On the right hand $\kappa x_1^2 \begin{vmatrix} x_2' \\ \xi_3 \end{vmatrix} \begin{vmatrix} x_2' \\ \xi_3' \end{vmatrix} = \dfrac{\kappa x_1^2 x_2' \kappa_3^2}{\rho^2}$.

*The result of this division is found in Klein's First Lecture, Winter Semester, 1889-90, on the "Nicht-Euclidsche Geometry," S. 40.

$$\therefore\ x_1^2 x_2'^2 \varkappa_3^2 = \frac{\varkappa}{\rho^2} x_1^2 x_2'^2 \varkappa_3^2. \quad \therefore\ \varkappa = \rho^2,$$

and we obtain the interesting result in determinants,*

$$\begin{vmatrix} \varkappa_1^2 & -\varkappa_1\varkappa_2\cos C & -\varkappa_1\varkappa_3\cos B & x_1 & x_1' \\ -\varkappa_1\varkappa_2\cos C & \varkappa_2^2 & -\varkappa_2\varkappa_3\cos A & x_2 & x_2' \\ -\varkappa_1\varkappa_3\cos B & -\varkappa_2\varkappa_3\cos A & \varkappa_3^2 & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{-iB} \\ x_2 & x_2' & \varkappa_2 e^{iA} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{iB} \\ x_2 & x_2' & \varkappa_2 e^{-iB} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \qquad (25)$$

D. The angle between the lines.

Let us take the four lines,

$$x+iy=0. \quad 1. \qquad x-iy=0. \quad 2.$$
$$x+\lambda y=0. \quad 3. \qquad x+\lambda' y=0. \quad 4.$$

The double ratio of these is, taking them in the order named using the ratio,

$$\alpha = \frac{(\mu_1-\mu_3)(\mu_4-\mu_2)}{(\mu_3-\mu_2)(\mu_1-\mu_4)},\ \dagger$$

where $\mu_1=i$, $\mu_2=-i$, $\mu_3=\lambda$, $\mu_4=\lambda'$, we have

$$\frac{(i-\lambda)(\lambda'+i)}{(\lambda+i)(i-\lambda')} = r+5i,$$

$$\text{or } i\lambda'-1-\lambda\lambda'-i\lambda = ri\lambda - r - r\lambda\lambda' - ri\lambda' - s\lambda - si - s\lambda\lambda' i + s\lambda.$$

Or, equating the real parts, and the imaginary parts, we have,

$$\lambda'-\lambda = r(\lambda-\lambda') - s(1+\lambda\lambda'),\ (\lambda-\lambda')(r+1) = s(1+\lambda\lambda').$$

$$1+\lambda\lambda' = r(1+\lambda\lambda') + s(\lambda-\lambda'),\ s(\lambda-\lambda') = (1-r)(1+\lambda\lambda').$$

$$\therefore\ \frac{\lambda-\lambda'}{1+\lambda\lambda'} = \frac{s}{1+r} = \frac{1-r}{s}.$$

And we see that r and s are restricted to the relation : $r^2+s^2=1$. If ϕ denote the angle between the lines 3 and 4,

*Compare Salmon's Conic Sections, page 133, Ex. 2.
†Clebsch, Vorlesungen ueber Geometrie, S. 38.

$$\tan\phi=\frac{s}{1+r}=\pm\sqrt{\frac{1-r}{1+r}}. \quad \therefore r=\cos 2\phi,\ s=\pm\sin 2\phi.$$

If we denote our double ratio of the four lines by (DR), and choose the lower sign for s, we have

$$(DR)=\cos 2\phi-i\sin 2\phi, \text{ or}$$
$$(DR)=e^{-2\phi i},$$
$$\phi=\tfrac{1}{2}i\log(DR). \quad (26)$$

We could have obtained this result in another way. If the equations of 3 and 4 were written more generally

$$ux+vy+1=0.$$
$$u'x+v'y+1=0.$$

$\tan\phi=\dfrac{uv'-u'v}{uu'+vv'}$. If $z=x+yi$, we have,

$$\log(x+yi)=\tfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}(y/x).$$
$$\log(x-yi)=\tfrac{1}{2}\log(x^2+y^2)-i\tan^{-1}(y/x).$$
$$\log\frac{x+yi}{x-yi}=2i\tan^{-1}(y/x)=2i\omega, \text{ where } \tan\omega=y/x.$$

$$\omega=\tfrac{1}{2}i\log\frac{x-yi}{x+yi}.$$

Using this as a formula for expressing ϕ, we get

$$\phi=\tfrac{1}{2}i\log\left(\frac{uu'+vv'+i(u'v-uv')}{uu'+vv'-i(u'v-uv')}\right)$$

$$=\tfrac{1}{2}i\log\left(\frac{uu'+vv'+\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}{uu'+vv'-\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}\right).* \quad (27)$$

But the expression under the logarithm is the quotient of the roots of the equation in λ,

$$u^2+v^2+2\lambda(uu'+vv')+\lambda^2(u'^2+v'^2)=0,\dagger$$

an equation obtained by substituting in the line equation of the circular points, (11), the values $u+\lambda u'$, $v+\lambda v'$, so that the ratio of the two roots of λ is again

*First given by Laguerre: Nouvelles Annales de Math. 1853.
†See Clebsch, Vorlesungen ueber Geometrie, S. 148.

the double ratio of the four lines. Klein in the before mentioned lecture on Non-Euclidean Geometry obtains the same result in still another way. The angle between two lines is thus also defined projectively with reference to the two fixed circular points at infinity, for the double ratio of four lines is an absolute invariant under linear transformation. Some special results in angle determination may interest us.*

1. The angle that a line to either of the two circular points makes with any other line of the finite region is to be regarded as infinite. For the tangent of that angle is given by

$$\frac{\tan\psi - i}{1 + i\tan\psi} = -i,$$

i, and ψ, being the tangents of the two lines. Now we have,

$$\tan x = -\frac{1}{i}\tanh xi = \frac{1}{i}\frac{e^{2xi}-1}{e^{2xi}+1}. \qquad \lim_{x=\infty}\tan x = \frac{1}{i}, \quad (\tan x)_{x=\infty} = -i.$$

The above angle between the two lines must therefore be regarded as an infinite one. Similarly for the line to the other circular point. By our new definition of angle, the matter is simpler still, for then in this case $\lambda = i$, or $-i$, and $(DR) = r + si = 0$, or ∞, whence $\phi = \infty$.

2. Two lines are perpendicular to each other when the double ratio (26) is equal to -1, that is when the four lines form a harmonic quadrupel. For using $-\pi i$ as $\log(-1)$ we get from (26), $\phi = \frac{1}{2}\pi$. Also above, put $r = -1$, $s = 0$, and obtain the same result.

3. Two lines are parallel when $r = 1$, $s = 0$, that is when the double ratio of the four lines is unity.

4. Two lines make an angle of 45°, or 135° when $r = 0$, $s = \pm 1$; that is when the double ratio is equal to $\pm i$.

5. All angles inscribed in a circle and intercepting the same arc are equal for the double ratio of four rays from some variable point in a circle to four fixed points in constant. Here the four fixed points are the two finitepoints at the ends of the arc, and the two fixed circular points at infinity. But if the double ratio is constant r and s are constant, therefore,

$$\frac{\lambda - \lambda'}{1 + \lambda\lambda'} = \frac{s}{1+r} = \frac{1-r}{s}$$

is constant, and the inscribed angle is constant.

IV. Relation of the Circular Points to Non-Euclidean Geometry.

What we have established in the preceding seems to suggest the way for investigations and generalizations of the greatest importance. And such was the course of history on the analytic side of the passage from Euclidean to Non-Euc-

†Clebsch, Vorlesungen ueber Geometrie, S. 147-149.

lidean geometry. It only remained to make the generalization that, $\Sigma xx = \Sigma a_{ik} x_i x_k = 0$, being the equation of the fundamental form in point or in line coördinates as might be needed, the expression

$$\kappa \log\left(\frac{\Sigma xx' + \sqrt{(\Sigma xx')^2 - \Sigma xx' . \Sigma x'x}}{\Sigma xx' - \sqrt{\Sigma xx' - \Sigma xx' . \Sigma x'x}} \right)$$

should be in general the distance between the points, or the angle between two lines. If $\kappa = \frac{1}{2}i$, and $\Sigma xx = u^2 + v^2$ we have the ordinary Euclidean angle between two lines. If $\Sigma xx'$ is not equal to $u^2 + v^2$, we evidently have something quite different from that angle, κ times the logarithm of the double ratio of the two lines and the pair of tangents to the conic from their point of intersection.

The derivation of the Euclidean distance formula is not so simple, a case of limits being involved. According as this fundamental conic is an actual one, a point pair, or an imaginary one, we get hyperbolic, parabolic, or elliptic metrical determination. Cayley seems to have given the first valuable suggestions tending towards analytic methods. Klein has built up an admirable analytic treatment, using what he calls the "Cayley'schen Maassbestimmung" as a basis. In his illustrations of elliptic, and hyperbolic geometry of the plane, he uses as fundamental conics $x^2 + y^2 = -r^2$, and $x^2 + y^2 = r^2$ respectively. It is interesting to note that the square of the element of length in each is,

$$ds_1^2 = \frac{dx^2 + dy^2 + \frac{(ydx - xdy)^2}{r^2}}{\left(1 + \frac{x^2 + y^2}{r^2}\right)^2}, \qquad ds_3^2 = \frac{dx^2 + dy^2 - \frac{(ydx - xdy)^2}{r^2}}{\left(1 - \frac{x^2 + y^2}{r^2}\right)^2}.$$

Now if r becomes indefinitely great, we have as the limit of both ds_1^2 and ds_3^2, $ds_2^2 = dx^2 + dy^2$, the square of the element of length in the ordinary Euclidean plane. This affords incidentally confirmation of our proposition under II. C, 2. that when the radius of a circle whose center is in the finite region becomes indefinitely great, the circle degenerates into the two circular points at infinity.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. Bonds are quoted in London at 108¾ and in Philadelphia at 112¼, exchange $4.48½, gold quoted at 107, how much more was a $1000 U. S. bond worth in London than in Philadelphia?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

If I understand the problem correctly, the exchange price is not necessary for the solution.

$\$1000 \times 1.12\frac{1}{4} = \$1122.50$, price in Philadelphia.

$\$1000 \times 1.08\frac{3}{4} = \$1087.50$, price in London.

But one dollar of London gold is worth $1.07 of Philadelphia currency.

$\therefore \$1087.50 \times 1.07 = \$1163.62\frac{1}{2}$, price of London bond in U. S. currency.

$\therefore \$1163.62\frac{1}{2} - \$1122.50 = \$41.12\frac{1}{2}$, the amount the London bond cost an American more than the Philadelphia bond.

To find the difference in cost to an Englishman in London, we proceed as follows:

$\$1000 \times 1.12\frac{1}{4} = \$1122.50$.

$\$1122.53 \div 1.07 = \$1049.06\frac{58}{107}$, price of the Philadelphia bond in English gold.

$\$1000 \times 1.08\frac{3}{4} = \$1087.50$.

$\$1087.50 - \$1049.06\frac{58}{107} = \$38.43\frac{49}{107}$.

$\$38.43\frac{49}{107} \div \$4.89\frac{1}{2} = 7£\ 17s\ .433d$.

[We believe Dr. Zerr's view of this problem to be the correct one. EDITOR.]

77. Proposed by F. S. ELDER, Professor of Mathematics, Oklahoma University, Norman, Oklahoma.

For how many seconds must I count the clicking of the rails under a train that the number of rails counted may be equal to the speed of the train in miles per hour, a rail being 30 feet long.

I. Solution by FREDERIC R. HONEY, Ph. B., New Haven, Connecticut, and CHAS. C. CROSS, Laytonsville, Maryland.

This problem is similar to the one proposed in the July-August number, Vol. III. The result is independent of the number of rails counted and of the number of miles per hour the train is running.

In the problem referred to, the answer is $3a/88$ minutes during which the poles are counted, where a equals the number of yards the polls are apart.

In the present case, $a=10$ yards. Hence, substituting, $3a/88$ minutes $=30/88$ minutes $=20\frac{5}{11}$ seconds.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and the PROPOSER.

Let t=number of seconds, n=number of miles per hour.

$\therefore$ $5280n/3600=22n/15$ feet per second=speed of train. Also in t seconds train goes $30n$ feet.

$\therefore$ $30n/t$=number of feet in one second.

$\therefore$ $30n/t=22n/15$. $\therefore$ $t=20\frac{5}{11}$ seconds.

78. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve by pure arithmetic, no algebraic symbols: A Texan farmer owns 5169 cattle; there are 3 times as many horses as cows, plus 569, and 4 times as many cows as sheep, minus 126; how many has he of each? [From *Brooks' Higher Arithmetic.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and J. C. CORBIN, Principal of Schools, Pine Bluff, Arkansas.

$5169+126-569=4726$=number of cattle when there are 4 times as many cows as sheep and 3 times as many horses as cows.

Every time he takes 1 sheep, he takes 4 cows and 12 horses, or 17 in all.

$\therefore$ he has as many lots of 1 sheep, 4 cows, 12 horses, as 17 is contained in 4726. $\therefore$ $4726\div17=278$.

$\therefore$ 278×1=number of sheep= 278

$278\times4-126$=number of cows= 986

$278\times12+569$=number of horses=3905

Total=5169

This problem was solved with a different view of its enunciation by Frederic R. Honey, and O. S. Westcott, A. M., Sc. D., Principal North Division High School, Chicago, Illinois.

[NOTE. P. S. Berg and H. C. Wilkes should each have received credit in the last number for solving problems 75 and 76. EDITOR.]

ALGEBRA.

Conducted by J. M, COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

70. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Given $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=\sqrt[3]{c}$ to find x.

I. Solution by J. MARCAS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Hewlett, Long Island, New York; EDWARD R. ROBBINS, Master in Mathematics and Physics in Lawrenceville School, Lawrenceville, New Jersey; E. L. SHERWOOD, A. M., Principal of City Schools, West Point, Mississippi; O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C.; A. H. HOLMES, Brunswick, Maine; and J. SCHEFFER, A. M., Hagerstown, Maryland.

Cubing, transposing, etc.,

$(a^2-x^2)^{\frac{1}{3}}\,[(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}]=(c-2a)/3$, or $(a^2-x^2)^{\frac{1}{3}}\,[\,c^{\frac{1}{3}}\,]=(c-2a)/3$.

$\therefore\ x=\pm\sqrt{\left(a^2-\frac{(c-2a)^3}{27c}\right)}$. If $c=2a$, $x=a$.

II. Solution by J. H. DRUMMOMD, LL. D., Portland, Maine; A. H. BELL, Hillsboro, Illinois; and CHAS. C. CROSS, Laytonsville, Maryland.

Transposing $\sqrt[3]{(a-x)}$, cubing and reducing, we have,

$$(a-x)^{\frac{1}{3}}-c^{\frac{1}{3}}(a-x)^{\frac{1}{3}}=(2a-c)/3c^{\frac{1}{3}}.$$

Completing the square, we find,

$$(a-x)^{\frac{1}{3}}=\tfrac{1}{2}\left[\ c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right].$$

Hence $x=a-\frac{1}{8}\left[\ c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right]^3$; or transposing $\sqrt[3]{(a+x)}$, and proceeding as before,

$$x=\tfrac{1}{8}\left[\ c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right]^3-a.$$

Messrs. Bell and Cross let $y=\sqrt[3]{(a+x)}$, substitute, and then solve as above.

III. Solution by H. C. WHITAKER, M. Sc., Ph. D., Professor of Mathematics in the Manual Training School, Philadelphia, Pennsylvania.

For convenience in writing, denote $\sqrt[3]{c}$ by b, $\sqrt[3]{(a+x)}$ by y and $\sqrt[3]{(a-x)}$ by z. Of the following equations, (1) is given and the others are assumed. (Take $\alpha^3=1$).

$y+z-b=0$ ..... (1).

$\alpha y+z-b=A$ .....(2).

$\alpha^2 y+z-b=B$ .....(3).

$y+\alpha z-b=C$ ..... (4).

$\alpha y+\alpha z-b=D$ .....(5).

$\alpha^2 y+\alpha z-b=E$ .....(6).

$y+\alpha^2 z-b=F$ .....(7).

$\alpha y+\alpha^2 z-b=G$ .....(8).

$\alpha^2 y+\alpha^2 z-b=H$ .....(9).

We have, by multiplying these three at a time,

$$[y^3+(z-b)^3][y^3+(\alpha z-b)^3][y^3+(\alpha^2 z-b)^3]=0.$$

Or, completing the multiplications,

$$(y^3+z^3-b^3)^3+27b^3y^3z^3=0.$$

Restoring the original values of y, z, and b, we get,

$$27c(x^2-a^2)=(2a-c)^3.$$

Hence $x=\sqrt{\left(\dfrac{(2a-c)^3+27a^2c}{27c}\right)}$.

This may be the root of the given equation or the root of any of the assumed equations, depending on the various values of a and c.

[This example is found in Bonnycastle's Algebra (1845), page 97.]
Also solved by *P. S. BERG*, *H. C. WILKES*, and *G. B. M. ZERR*.

71. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

When $x=0$, find the limit of the expression

$$u=\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}+\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}.$$

I. Solution by O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C., and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $u=u_1+u_2$. $\therefore\ u_1=\left(\dfrac{m+x}{m-x}\right)^{\frac{1}{x}}$,

$$\log u_1=(1/x)\{\log(m+x)-\log(m-x)\}$$

$$=(1/x)\{[\log m+(x/m)-(x^2/2m^2)+(x^3/3m^3)-\ldots\ldots\ ]-$$

$$[\log m-(x/m)-(x^2/2m^2)-(x^3/3m^3)-\ldots\ldots\ ]\}$$

$$=\frac{2}{x}\left(\frac{x}{m}+\frac{x^3}{3m^3}+\frac{x^5}{5m^5}+\ldots\ldots\right)=2\left(\frac{1}{m}+\frac{x^2}{3m^2}+\frac{x^4}{5m^5}+\ldots\ldots\right)$$

$=2/m$, when $x=0$. $\log u_2=-\log u_1=-2/m$, when $x=0$.

$\therefore\ u_1=e^{2/m}$, $u_2=e^{-2/m}$. $\therefore\ u=e^{2/m}+e^{-2/m}$ when $x=0$.

II. Solution by H. C. WHITAKER, M. Sc., Ph. D., Professor of Mathematics in Philadelphia Manual Training School, Philadelphia, Pennsylvania.

Since $(1+x)^{1/x}=e$ when $x=0$, we have

$$\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}=\left[\left(1+\frac{2x}{m-x}\right)^{\frac{m-x}{2x}}\right]^{\frac{2}{m-x}}=e^{2/m} \text{ when } x=0.$$

In the same way $\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}=e^{-2m}$ when $x=0$. Hence $u=e^{2m}+e^{-2m}$.

Also solved by *J. SCHEFFER*.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

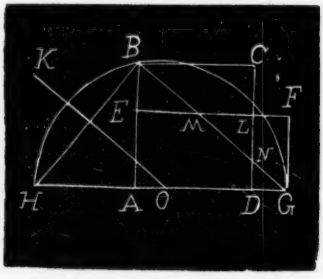
SOLUTIONS OF PROBLEMS.

69. Proposed by WILLIAM SYMMONDS, M. A., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To divide a square card into right-lined sections in a manner, that a rectangle of a given breadth can be formed from the sections; likewise, form a square from a rectangular card.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

(1). Let $ABCD$ be the square. Produce DA to H making AH equal the given width of the rectangle, join HB, and draw KO perpendicular to HB at its mid-point, then O is the center of the circle through HB. Produce AD to meet circle at G; AG is the length of the required rectangle. Take $AE=AH$ and complete the rectangle $AEFG$.



Now the right triangle AHB=right triangle BCN=right triangle MFG.

$\therefore$ $CN=AE$ and $DN=BE$;

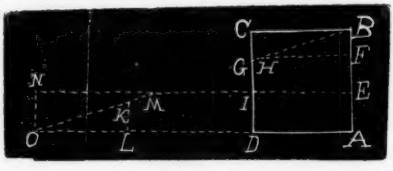
$\therefore$ $\triangle BEM=\triangle DNG$.

$\therefore$ $ABCD=ADNME+BCN+BEM$
$=ADNME+MFG+NDG=AEFG$.

(2). Let $AEFG$ be the given rectangle. Produce GA to H making $AH=AE$. Upon HG describe the semi-circle. Then AB is a side of the required square. Complete the square $ABCD$ and draw BG. The rest of the proof is the same as above.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1). Let $ABCD$ represent the square card. From A lay off on AB the the width of the rectangle successively as many times as possible, as AE, EF.



Then from the opposite corner C, lay off *one width only* of the rectangle, as CG. Now cut through on line GB. Then cut FH and EI parallel to DA.

Then will $EFGHI$ coincide with $DIMKL$, FBH with LKO, and BCG with MNO; and we have the rectangle $AENO$, of the given width AE, equivalent to the square $ABCD$.

(2). Let $AENO$ represent a rectangular card. Find the side of a square equivalent to the given rectangle. (Any geometry will show construction.)

Now from A lay off on OA the side of the square successively as many times as possible, as AD, DL. From N, lay off $NM=AD$.

Now, cut through on line OM. Then cut LK and DI perpendicular to AO.

Then will the sections of rectangle form a square as shown in first part of solution. The proof is evident.

III. Solution by C. H. WILSON, New York, New York.

Let $ABCD$ be the given square, and EF the breadth of the required rectangle. Find GH, a third proportional to EF and AB.

With C as a center and radius equal to GH, describe an arc cutting AB at K. (If $GH<BC$, use EF as radius). Cut off the triangle CBK and attach it in the position DAL. Draw KN perpendicular to LD. Cut of $\triangle LKN$ and attach it in the position DCM. $NKCM$ is the required rectangle, since it is equivalent to the square, and its area equals $NM\times NK=GH\times NK$. But by construction $GH\times EF=\overline{AB}^2$. $\therefore$ $EF=NK$ (Ax).

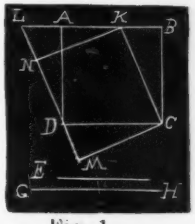


Fig. 1.

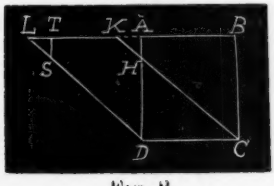


Fig. 2.

If K falls beyond A, attach $ARCB$ in position $TSDA$, cut off KAR and attach it in position LTS.

Then proceed as before.

Reverse operation (Fig. 1.) by finding mean proportional between MN and MC. With it as a radius and C as a center, describe an arc cutting MN in D. Cut off $\triangle MDC$ and attach it as NLK. Draw DA perpendicular to KL, cut off $\triangle LAD$ and attach it as KBC.

IV. Solution by the PROPOSER.

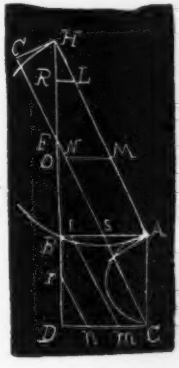
On AC, a side of the given square $ABCD$, draw a semi-circle AEC. Measure from A the chord AE equal to the given breadth of the rectangle. Produce CE, marking AB in S. On CD take CM, MN, etc., equal to AS.

Through m, n, etc., draw ml, nr, etc., parallel to CS. These will trace the lines of division required.

From the figure it is plain the rectangle $AEGH$ can be formed by joining fragments, AEC, AES, SM, m, Drn.

(2). On one of the longer sides AH of the given rectangle $AEGH$ describe the semi-circle ABH.

From A lay off chord AB equal in length to side of required square.

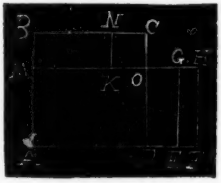


Join BH. Take BO, OR, etc., equal to AB. Through O, R, etc., draw OM, RL, etc., parallel to BA, marking the lines of division MN, LR, etc.

Hence the square $ABCD$ forms the parts of the rectangle, AES, SM, MR, RLH, HGF.

COROLLARY. When AS is greater than AB, or conversely, when AB is less than AS, the construction is quite simple.

V. Solution by J. W. SCROGGS, Principal of Rogers Academy, Rogers, Arkansas.



Let $ABCD$ be the given square, $AB=a+b$, $AM=MK=OD=GE=a$, $OK=OC=DE=b$, and $EF=b/r$.

Then area of $ADOM=a^2+ab$, and area of $BNKM=ab$.

$\therefore$ Their sum$=a^2+2ab$.

Let $b/a=r$. Then $a=br$, and area of $EFHG=b/r\times br=b^2$. $\therefore$ $AFHM=ABCD$.

CALCULUS.

Conducted by J. M, COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscata, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscata whose axis b the axis of a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

CASE I.

The equation of strophoid with origin at node is,

$$y^2=\frac{x(x-a)^2}{2a-x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation of lemniscate with origin at crunode is,

$$(x^2+y^2)^2=b^2(x^2-y^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In order that the latter may be inscribed in the former we must have tangency. $\therefore$ x, y, and dy/dx must be equal for each curve.

Substituting y from (1) in (2) we get after reduction,

$$x^3-4ax^2+\left(\frac{9a^2b^2-a^4}{2b^2}\right)x-a^3=0 \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots(3).$$

$$\frac{dy}{dx}=\frac{b^2x-2x^3-2xy^2}{y(b^2+2x^2+2y^2)}=\frac{4ax^2-4a^2x+a^3-x^3}{y(2a-x)^2} \quad \ldots\ldots\ldots\ldots(4).$$

Substituting y from (1) in (4) and reducing we get,

$$x^4-6ax^3+12a^2x^2-\left(\frac{17a^3b^2-2a^5}{2b^2}\right)x+a^4=0\ldots\ldots\ldots\ldots(5).$$

(5) may be written as follows :

$$x^4-5ax^3+8a^2x^2-\left(\frac{8a^3b^2-a^5}{2b^2}\right)x-a\{x^3-4ax^2+\left(\frac{9a^2b^2-a^4}{2b^2}\right)x-a^3\}=0.$$

(3) in the last equation gives,

$$x^3-5ax^2+8a^2x-\frac{8a^3b^2-a^5}{2b^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

(3)$-$(6) gives, $x^2-\frac{7ab^2+a^3}{2b^2}x-\frac{a^4-6a^2b^2}{2b^2}=0$ $\ldots\ldots\ldots\ldots\ldots\ldots(7).$

$\frac{8b^2-a^2}{2b^2}$ times (3)$-$(6) gives,

$$(6b^2-a^2)x^2-(22ab^2-4a^3)x+\frac{40a^2b^4-17a^4b^2+a^6}{2b^2}=0 \quad \ldots\ldots\ldots(8).$$

$(6b^2-a^2)$ times (7)$-$(8) gives,

$$x=\frac{4ab^4-5a^3b^2}{2b^4-7a^2b^2+a^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

$\frac{40a^2b^4-17a^4b^2+a^6}{2b^2}$ times (7)$+\frac{a^4-6a^2b^2}{2b^2}$ times (8) gives,

$$x=\frac{16ab^6+13a^3b^4-18a^5b^2+a^7}{8b^6-10a^2b^4} \quad \ldots\ldots\ldots\ldots\ldots(10).$$

From (9) and (10) we get,

$$6b^8+161a^2b^6-141a^4b^4+25a^6b^2-a^8=0\ldots\ldots\ldots\ldots(11).$$

Let $a^2/b^2=u$, then (11) becomes,

$$u^4-25u^3+141u^2-161u-6=0 \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

$\therefore u=1.586892$. $\therefore a^2=1.586892b^2$.

$\therefore a=1.2597b$. $\therefore b=.7938a$.

CASE II.

The equation of the strophoid with origin at crunode is,

$$y^2=\frac{x^2(a+x)}{a-x} \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The equation of the lemniscate with origin at node is,

$$\{(x+b)^2+y^2\}^2=b^2\{(x+b)^2-y^2\} \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

In order that the former may be inscribed in the latter we must have x, y, dy/dx equal for both curves.

(13) in (14) gives,

$$(2a^2+b^2-4ab)x^3+(4a^2b-5ab^2+b^3)x^2+(4a^2b^2-2ab^3)x+a^2b^3=0 \ldots\ldots\ldots (15).$$

$$\frac{dy}{dx}=\frac{a^2x+ax^2-x^3}{y(a-x)^2}=-\frac{2x^3+6bx^2+5b^2x+b^3+2(x+b)y^2}{y(2x^2+4bx+3b^2+2y^2)} \ldots\ldots (16).$$

(13) in (16) gives,

$$(8ab-4a^2-2b^2)x^4+(8a^3-20a^2b+9ab^2-b^3)x^3$$
$$+(12a^3b-15a^2b^2+3ab^3)x^2+(8a^3b^2-3a^2b^3)x+a^3b^3=0 \ldots\ldots (17).$$

(17) may be written

$$(8ab-4a^2-2b^2)x^4+(6a^3-16a^2b+8ab^2-b^3)x^3$$
$$+(8a^3b-10a^2b^2+2ab^3)x^2+(4a^3b^2-a^2b^3)x$$
$$+a[(2a^2+b^2-4ab)x^3+(4a^2b-5ab^2+b^3)x^2+(4a^2b^2-2ab^3)x+a^2b^3]=0 \ldots (18).$$

(15) in (18) gives,

$$2(2a^2+b^2-4ab)x^3+(16a^2b-8ab^2+b^3-6a^3)x^2$$
$$+(10a^2b^2-8a^3b-2ab^3)x+a^2b^3-4a^3b^2=0 \ldots\ldots (19).$$

(19)−2 times (15) gives,

$$(8a^2b+2ab^2-b^3-6a^3)x^2+(2a^2b^2-8a^3b+2ab^3)x-a^2b^3-4a^3b^2=0 \quad \ldots\ldots(20).$$

b times (19)−$(b-4a)$ times (15) gives,

$$(8a^3+b^3-14a^2b)x^2+(ab^3+10a^3b-8a^2b^2)x+8a^3b^2-2a^2b^3=0 \quad \ldots\ldots\ldots(21).$$

$(8a^3+b^3-14a^2b)$ times (20)−$(8a^2b+2ab^2-b^3-6a^3)$ times (21) gives,

$$x=\frac{14a^3b^3+16a^5b+8a^2b^4-28a^4b^2-3ab^5}{4a^5-38a^3b^2+8ab^4+18a^2b^3-3b^5} \quad \ldots\ldots\ldots(22).$$

$(8a-2b)$ times (20)+$(4a+b)$ times (21) gives,

$$x=\frac{10a^3b^2-24a^4b+8a^2b^3-3ab^4}{14a^2b^2+16a^4+8ab^3-28a^3b-3b^4} \quad \ldots\ldots\ldots\ldots(23).$$

From (22) and (23) we get,

$$9b^5+16ab^4-148a^2b^3-144a^3b^2+468a^4b-176a^5=0\ldots\ldots\ldots(24).$$

Let $b/a=v$, then (24) becomes,

$$9v^5+16v^4-148v^3-144v^2+468v-176=0.$$

$\therefore v=1.12257$. $\therefore b=1.12257a$. $\therefore a=.8908b$.

The published solution assumes that both curves coincide at their crunodes. This is not what the problem calls for. The above solution realizes in every respect the demands of the problem.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What is the position of equilibrium of the center?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $BC=a$, $AC=b$, $AB=c$, $AP=h$, $BP=k$, $CP=l$, $AL=m$, $PL=n$, $\angle APB=\beta$, $\angle BPC=\gamma$, $\angle APC=\delta$.

Then $\beta=\cos^{-1}\left(\frac{k^2+h^2-c^2}{2hk}\right)$, $\gamma=\cos^{-1}\left(\frac{k^2+l^2-a^2}{2kl}\right)$, $\delta=\cos^{-1}\left(\frac{h^2+l^2-b^2}{2hl}\right)$,

$$h^2=m^2+n^2,\ k^2=(c-m)^2+n^2,\ l^2=(b\cos A-m)^2+(b\sin A-n)^2.$$

If P be the required point, A origin, AX, AY, axes, $AM=x$, ρ=density and $\varkappa$ area of cross section of the lines perpendicular to their length, ν=mass of P, μ=some constant, then the attraction on M is,

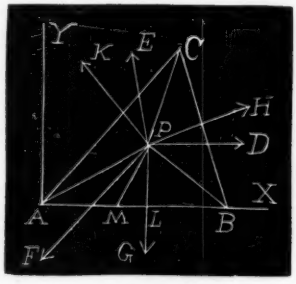
$$\frac{\mu\nu\rho\varkappa dx}{PM}=\frac{\mu\nu\rho\varkappa dx}{\{n^2+(m-x)^2\}^{\frac{1}{2}}}.$$

Call $\mu\nu\rho\varkappa$, Q. Then the resolved part of the force parallel to axis of X is,

$$\frac{Q(m-x)dx}{n^2+(m-x)^2},$$

and the resolved part parallel to the axis of Y is,

$$\frac{Qndx}{n^2+(m-x)^2}.$$



$$\therefore\ PD=-Q\int_0^c\frac{(m-x)dx}{n^2+(m-x)^2}=\tfrac{1}{2}Q\log\left\{\frac{n^2+(c-m)^2}{n^2+m^2}\right\}.$$

$$\therefore\ PD=Q\log(k/h),\ PE=Q\log(l/k),\ PF=Q\log(h/l).$$

$$PG=-Q\int_0^c\frac{ndx}{n^2+(m-x)^2}=-Q\{\tan^{-1}(m/n)+\tan^{-1}\frac{c-m}{n}\}.$$

$$\therefore\ PG=-Q\beta,\ PH=-Q\gamma,\ PK=-Q\delta.$$

If the resultant of two forces is equal to the third, the three forces are in equilibrium.

Regarding PD, PE, PF as three forces and PG, PH, PK as three forces, we get,

$$\left.\begin{aligned}&\{\log(k/h)\}^2+\{\log(l/k)\}^2+2\log(k/h)\log(l/k)\cos B=\{\log(h/l)\}^2\\&\beta^2+\gamma^2-2\beta\gamma\sin B=\delta^2\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned}&\{\log(l/k)\}^2+\{\log(h/l)\}^2+2\log(l/k)\log(h/l)\cos C=\{\log(k/h)\}^2\\&\gamma^2+\delta^2-2\gamma\delta\sin C=\beta^2\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{aligned}&\{\log(h/l)\}^2+\{\log(k/h)\}^2+2\log(h/l)\log(k/h)\cos A=\{\log(l/k)\}^2\\&\beta^2+\delta^2-2\beta\delta\sin A=\gamma^2\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(3).$$

Substituting the values of h, k, l, β, γ, δ in terms of m, n in either (1), (2), or (3), we get, in either case, two equations in m and n from which, if possible, the values of m and n may be found and the point P determined.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

80. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

From a cask containing 10 gallons of wine, a servant drew off 1 gallon each day, for five days, each time supplying the deficiency by adding a gallon of water. Afterwards, fearing detection, he again drew off a gallon a day for five days, adding each time a gallon of wine. How many gallons of water still remained in the cask? [From *Quackenbos' Arithmetic.*]

81. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

How far will a body fall in the first second on the sun, the density of the sun being 25 times that of the earth and its diameter 866400 miles?

82. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

Two men, A and B, started from the same point at the same time; A traveled southeast for 10 hours and at the rate of 10 miles per hour, and B due south for the same time, going 6 miles per hour; they then turned and traveled directly towards each other at the same rates respectively, till they met. How far did each man travel?

DIOPHANTINE ANALYSIS.

55. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Construct a general Magic Square whose sum is $3m$.

56. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

If $\phi(R)$ is the number of integers which are less than R and prime to it, and if y is prime to R, show that $y^{\phi(R)}-1\equiv 0(\mathrm{mod}.R)$.

57. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Each of *five* of the digits may be the terminal figure of a perfect integral square. Each of *eighteen* combinations of two digits may be the *two* terminal figures of an integral square. Each of *one hundred and nineteen* combinations of three digits may be the *three* terminal figures of an integral square. *Under these conditions*, what is the greatest number of arrangements of the nine digits, all taken together, whose three terminal figures shall be those of a square number?

MISCELLANEOUS.

53. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

(a). What is the highest north latitude in which the sun will shine in at the north window of a building at least once in a year?

(b). How many days will it shine in at the north window of a building in latitude 41° N.?

54. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

On latitude 40° N.$=\gamma$, when the moon's declination is 5° 23′ N.$=\delta$, and the sun's 9° 52′ S.$=-\delta$, how long after sunset, will the two horns or cusps of the moon's crescent (recently new) set at the same moment, the crescent with its back *down* having touched the horizon first? Semi-diameters, refraction, and parallax not considered.

55. Proposed by J. M. COLAW, A. M., Monterey, Virginia.

Multiply 6 by 4. Is the problem legitimate when both symbols represent pure number?

[NOTE. "A measured or numbered quantity may be divided into a number of parts, or taken a number of times; but no number can be multiplied or divided into parts."—*McLellan and Dewey's Psychology of Number.* "The astounding thesis is maintained that number is not a magnitude, does not possess quantity at all, and that 'no number can be multiplied or divided into parts'."—*Lefevre's Number and Its Algebra.*]

BOOKS.

Theory of Discrete Manifoldness. By F. W. Franklin. Pamphlet, 12 pages. Published by the Author.

Recent Books on Quaternions (from "Science," Vol. V, pages 699—701). By Dr. Alexander Macfarlane.

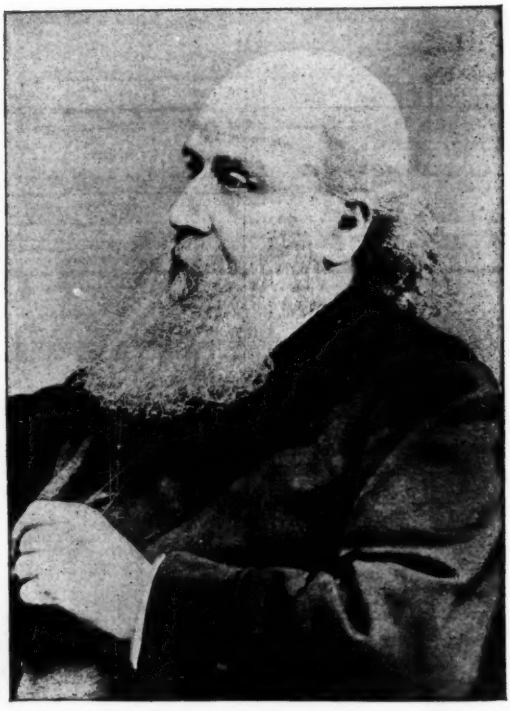
This is a concise criticism on the following works: *Theorie der Quaternionen*, Von Dr. P. Molenbroke; *Anwendung der Quaternionen auf die Geometrie*, by same author; *The Outlines of Quaternions*, by Lieut. Col. H. W. L. Hime; *A Primer of Quaternions*, by A. S. Hathaway; and *Utility of Quaternions in Physics*, by A. McAulay. B. F. F.

Application of Hyberbolic Analysis to the Discharge of a Condenser. By Dr. Alexander Macfarlane. Pamphlet, 16 pages.

A paper presented at the annual meeting of the American Institute of Electrical Engineers, May 18th, 1897. B. F. F.

Introduction to American Literature, including Illustrative Selections with Notes. By F. V. N. Painter, A. M., D. D., Professor of Modern Languages in Roanoke College, Author of a History of Education, Introduction to English Literature, etc. 8vo. Cloth, 498 pages. Boston: Leach, Shewell and Sanborn.

This is the best prepared work on American Literature with which we are acquainted. Only the very best productions from the best authors are selected. The selections for special notice, which are chosen to illustrate the distinguishing characteristics of each author are supplied with explanatory notes. The short sketch of our leading writers is written in an intensely interesting manner, all the kernels having been preserved and all the chaff thrown away. Each sketch is preceded by an excellent portrait. B. F. F.



JAMES JOSEPH SYLVESTER.